NORTH-HOLLAND

Invertible Preservers of Certain Decomposable Tensors

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ABSTRACT

Let U be an n-dimensional vector space over a field of characteristic 0. For each positive integer $k < \min\{m, n\}$, let J_k be the set of all decomposable elements $x_1 \otimes \cdots \otimes x_m$ in the mth tensor product $\overset{m}{\otimes} U$ such that $\dim\langle x_1, \ldots, x_m \rangle \leqslant k$. We characterize those nonsingular linear mappings T on $\overset{m}{\otimes} U$ such that $T(J_k) \subseteq J_k$.
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1. INTRODUCTION

Let F be a field and $M_n(F)$ be the vector space of all $n \times n$ matrices over F. In [3], Dieudonné showed that if T is a nonsingular linear mapping on $M_n(F)$ that sends the set of singular matrices into itself, then either $T(A) = PAQ$ or $T(A) = PA^tQ$ for some nonsingular matrices P and Q in $M_n(F)$. Dieudonné's result implies that if T is a nonsingular linear mapping on $M_n(F)$ that sends the set of all rank one matrices into itself, then T is of the above-mentioned form.

Let U be a finite dimensional vector space over F. For each σ in the symmetric group S_m of degree m, let P_σ denote the linear mapping on the mth tensor space $\overset{m}{\otimes} U$ such that

$$P_\sigma(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}$$

LINEAR ALGEBRA AND ITS APPLICATIONS 266:153–159 (1997)

0024-3795/97/\$17.00
PII S0024-3795(96)00642-8

for all $x_1, \ldots, x_m$ in U. In [9], Westwick obtained the following theorem that generalizes the result concerning nonsingular rank one preserves on $M_n(F)$.

THEOREM 1. *Let $T : \overset{m}{\otimes} U \to \overset{m}{\otimes} U$ be a linear mapping sending nonzero decomposable elements to nonzero decomposable elements. If* (i) *T is nonsingular or* (ii) *F is algebraically closed, then*

$$T = P_\sigma \circ \bigotimes_{i=1}^{m} f_i$$

for some σ in S_m and some nonsingular linear mappings f_i on U, $i = 1, \ldots, m$.

Case (ii) of Theorem 1 was proved for $m = 2$ and char $F = 0$ by Marcus and Moyls [6].

Let P be the subset of all decomposable tensors in $\overset{m}{\otimes} U$ of the form $x \otimes \cdots \otimes x$. In [8], Shaw proved the following result:

THEOREM 2. *Let T be a nonsingular linear mapping on $\langle P \rangle$ such that $T(P) \subseteq P$. If* char $F = 0$, *then*

$$T = \overset{m}{\otimes} f|_{\langle P \rangle}$$

for some nonsingular linear mapping f on U.

Let dim $U = n$. For each positive integer $k \leqslant \min\{n, m\}$, let

$$D_k = \{x_1 \otimes \cdots \otimes x_m : \dim\langle x_1, \ldots, x_m \rangle = k\} - \{0\},$$

$$J_k = \{x_1 \otimes \cdots \otimes x_m : \dim\langle x_1, \ldots, x_m \rangle \leqslant k\}.$$

Notice that J_k is the set of all decomposable elements in $\overset{m}{\otimes} U$ if $k = \min\{n, m\}$. It is easily shown that $\langle J_s \rangle \subsetneqq \langle J_{s+1} \rangle$ for $s < \min\{n, m\}$. Notice that $\langle J_1 \rangle = \langle P \rangle$ and $\langle P \rangle$ is the subspace of all symmetric tensors in $\overset{m}{\otimes} U$ when char $F = 0$. It can be shown that J_k is the intersection of $\langle J_k \rangle$ and the set of all decomposable tensors in $\overset{m}{\otimes} U$. Hence J_k is an algebraic variety of $\overset{m}{\otimes} U$.

The purpose of this note is to prove the following result:

THEOREM 3. *Let F be a field of characteristic* 0. *Let k be a fixed positive integer such that $k < \min\{n, m\}$. Let T be a linear mapping on $\langle J_k \rangle$. Suppose* *one of the following conditions holds*:

(a) *T is nonsingular and* $T(J_k) \subseteq J_k$;
(b) *T is nonsingular and* $T(D_k) \subseteq D_k$;
(c) $T(J_k - \{0\}) \subseteq J_k - \{0\}$ *and F is algebraically closed.*

Then there exist λ in $F - \{0\}$, σ in S_m, and some nonsingular linear mapping f on U such that

$$T(x_1 \otimes \cdots \otimes x_m) = \lambda f(x_{\sigma(1)}) \otimes \cdots \otimes f(x_{\sigma(m)})$$

for all $x_1 \otimes \cdots \otimes x_m \in J_k$.

A subspace of $\overset{m}{\otimes} U$ is called a decomposable subspace if it consists entirely of decomposable elements in $\overset{m}{\otimes} U$. It follows from Lemma 3.1 in [9] that every decomposable subspace of $\overset{m}{\otimes} U$ is of the form

$$u_1 \otimes \cdots \otimes u_{i-1} \otimes W \otimes u_{i+1} \otimes \cdots \otimes u_m$$

for some i, some vectors $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_m$ in U, and some subspace W of U. Let $y_1 \otimes \cdots \otimes y_m \in J_k$, where $y_1, y_2, \dots, y_k$ are linearly independent and $k < \min\{n, m\}$. Then it is easily seen that every n-dimensional decomposable subspace contained in J_k and containing $y_1 \otimes \cdots \otimes y_m$ is of the following form:

$$y_1 \otimes \cdots \otimes y_{i-1} \otimes U \otimes y_{i+1} \otimes \cdots \otimes y_k \otimes \cdots \otimes y_m$$

for some i such that $1 \leqslant i \leqslant k$. This fact will be used in the proof of Theorem 3.

2. PROOF OF THEOREM 3

Proof of part (a). Case 1: $k = 1$. Let $V = U \otimes_F \bar{k}$, where $\bar{k}$ is the algebraic closure of F. We identify U as an F-subspace of V via the F-linear mapping $u \to u \otimes 1$, and $\overset{m}{\otimes} U$ as an F-subspace of $\overset{m}{\otimes} V$ via the F-linear mapping sending $u_1 \otimes \cdots \otimes u_m$ to $(u_1 \otimes 1) \otimes \cdots \otimes (u_m \otimes 1)$. Now regard V and $\overset{m}{\otimes} V$ as vector spaces over $\bar{k}$. Let T_1 be any linear mapping on $\overset{m}{\otimes} U$ such that $T = T_1|_{\langle J_k \rangle}$. Let S be the extension mapping of T_1 to $\overset{m}{\otimes} V$. Since $T(J_1) \subseteq J_1$, it follows that

$$S(\bar{J}_1) \subseteq \bar{J}_1$$

where $\bar{J}_1$ denotes the Zariski closure of J_1 in $\overset{m}{\otimes} V$. Note that

$$\bar{J}_1 = \{ y \otimes \cdots \otimes y : y \in V \}.$$

In view of Theorem 2,

$$S(y \otimes \cdots \otimes y) = g(y) \otimes \cdots \otimes g(y), \qquad y \in V,$$

for some nonsingular linear mapping g on V. Let $e_1, \dots, e_n$ be a basis of U. Then

$$T(e_i \otimes \cdots \otimes e_i) = g(e_i) \otimes \cdots \otimes g(e_i) = \lambda_i x_i \otimes \cdots \otimes x_i$$

for some $\lambda_i \in F$, $x_i \in U$. Hence

$$g(e_i) = a_i x_i \qquad \text{for some} \quad a_i \in k$$

with $a_i^m = \lambda_i$, $i = 1, \dots, m$. Let $e = \sum_{i=1}^n e_i$. Then

$$\begin{aligned} T(e \otimes \cdots \otimes e) &= g(e) \otimes \cdots \otimes g(e) \\ &= \left(\sum_{i=1}^n a_i x_i \right) \otimes \cdots \otimes \left(\sum_{i=1}^n a_i x_i \right) \\ &= \lambda u \otimes \cdots \otimes u \end{aligned}$$

for some $\lambda \in F$, $u \in U$. Let $u = \sum_{i=1}^n c_i x_i$, $c_i \in F$. Then

$$a_{i_1} a_{i_2} \cdots a_{i_m} = \lambda c_{i_1} c_{i_2} \cdots c_{i_m} \qquad \text{for any} \quad 1 \leqslant i_j \leqslant n.$$

Define a linear mapping f on U as follows:

$$f(e_i) = c_i c_1^{-1} x_i, \qquad i = 1, \dots, n.$$

Then for any $v = \sum_{i=1}^n b_i e_i \in U$ where $b_i \in F$,

$$\begin{aligned} \left(\lambda_1 \overset{m}{\otimes} f\right)(v \otimes \cdots \otimes v) &= \lambda_1 \sum b_{i_1} \cdots b_{i_m} c_{i_1} c_1^{-1} \cdots c_{i_m} c_1^{-1} x_{i_1} \otimes \cdots \otimes x_{i_m} \\ &= \sum b_{i_1} \cdots b_{i_m} a_{i_1} \cdots a_{i_m} x_{i_1} \otimes \cdots \otimes x_{i_m} \\ &\quad (\because \lambda_1 = a_1^m = \lambda c_1^m) \\ &= T(v \otimes \cdots \otimes v). \end{aligned}$$

This proves case 1.

Case 2: $k \geqslant 2$. We shall show that $T(J_{k-1}) \subseteq J_{k-1}$. Suppose that there exists $A = x_1 \otimes \cdots \otimes x_m \in J_{k-1} - \{0\}$ such that

$$T(A) = y_1 \otimes \cdots \otimes y_m \in D_k.$$

Without loss of generality, we may assume that $y_1, \ldots, y_k$ are linear independent. Let

$$Z_i = x_1 \otimes \cdots \otimes x_{i-1} \otimes U \otimes x_{i+1} \otimes \cdots \otimes x_m, \qquad i = 1, \ldots, m.$$

Then $Z_i \subseteq J_k$, and hence $T(Z_i)$ is an n-dimensional decomposable subspace contained in J_k and $T(A) \in T(Z_i)$. Since $k < \dim U$, it follows that

$$T(Z_i) = y_1 \otimes \cdots \otimes y_{s_i - 1} \otimes U \otimes y_{s_i + 1} \otimes \cdots \otimes y_k \otimes \cdots \otimes y_m$$

for some $1 \leqslant s_i \leqslant k$. This is a contradiction, since

$$T(Z_i) \neq T(Z_j) \qquad \text{for distinct } i \text{ and } j.$$

Hence $T(J_{k-1}) \subseteq J_{k-1}$. By induction, we obtain that $T(J_1) \subseteq J_1$. Thus

$$T(x \otimes \cdots \otimes x) = af(x) \otimes \cdots \otimes f(x), \qquad x \in U,$$

for some $a \in F - \{0\}$ and some nonsingular linear mapping f on U.

Let W be any k-dimensional subspace of U. Let L be the restriction map of T to $\overset{m}{\otimes} W$. Then L is an injective linear mapping from $\overset{m}{\otimes} W$ to $\overset{m}{\otimes} U$ sending decomposable elements to decomposable elements. In view of Theorem 1 in [9], either

$$\text{(i)} \qquad L = P_\sigma \circ \bigotimes_{i=1}^{m} f_i$$

for some σ in S_m and some nonsingular linear mappings $f_1 : W \to U$, or

$$\text{(ii)} \qquad L = P_\sigma \circ M_y \circ \left(\bigotimes_{i=1}^{s} g_1 \right) \circ P_\tau$$

for some $\sigma, \tau \in S_m$, some linear mappings $g_i : \overset{m_i}{\otimes} W \to U$, $i = 1, \ldots, s$, and some decomposable element $y = y_1 \otimes \cdots \otimes y_t$ in $\overset{t}{\otimes} U$, where M_y is the linear mapping from $\overset{m-t}{\otimes} U$ to $\overset{m}{\otimes} U$ such that

$$M_y(u_1 \otimes \cdots \otimes u_{m-t}) = y \otimes u_1 \otimes \cdots \otimes u_{m-t}.$$

Case (ii) is clearly not possible, since L sends J_1 to J_1. Hence (i) holds and we have

$$\begin{aligned} L(x \otimes \cdots \otimes x) &= f_{\sigma^{-1}(1)}(x) \otimes \cdots \otimes f_{\sigma^{-1}(m)}(x) \\ &= af(x) \otimes \cdots \otimes f(x) \neq 0 \qquad \text{for any} \quad x \in W - \{0\}. \end{aligned}$$

Hence $\langle f_i(x)\rangle = \langle f(x)\rangle$ for any $x \in W$. Therefore $f_i = d_i f|_W$ for some $d_i \in F$. This proves that $d_1 d_2 \cdots d_m = a$ and hence

$$L(x_1 \otimes \cdots \otimes x_m) = af(x_1) \otimes \cdots \otimes f(x_m) \qquad \ldots(1)$$

for any $x_1 \otimes \cdots \otimes x_m \in \overset{m}{\otimes} W$. Since W is any arbitrary k-dimensional subspace, (1) is true for all $x_1 \otimes \cdots \otimes x_m \in J_k$. This completes the proof. ∎

Proof of part (b). Let $H_k = \{(v_1, \ldots, v_m): v_i \in U, \dim\langle v_1, \ldots, v_m\rangle \leqslant k\}$ and $G_k = \{(v_1, \ldots, v_m): v_i \in U, \dim\langle v_1, \ldots, v_m\rangle = k\}$. Then it is known that $\overline{G}_k = H_k$, where $\overline{M}$ denotes the Zariski closure of M. Let θ be the mapping from $\overset{m}{\times} U$ to $\overset{m}{\otimes} U$ such that

$$\theta(v_1, \ldots, v_m) = v_1 \otimes \cdots \otimes v_m.$$

Since $\theta(G_k) = D_k$ and θ is a morphism, it follows that

$$\theta(\overline{G}_k) \subseteq \overline{D}_k.$$

This proves that $\theta(H_k) = J_k \subseteq \overline{D}_k$. Since J_k is a variety containing D_k, it follows that $\overline{D}_k = J_k$. From the hypothesis $T(D_k) \subseteq D_k$, we have $T(\overline{D}_k) \subseteq \overline{D}_k$. Hence the result follows from part (a). ∎

Proof of part (c). Since H_k is an irreducible variety and $\theta(H_k) = J_k$, it follows that J_k is a homogeneous irreducible variety. Since $T(J_k - \{0\}) \subseteq J_k - \{0\}$, it follows from Lemma 3 in [1] that $T|_{\langle J_k\rangle}$ is a nonsingular linear mapping on $\langle J_k\rangle$. Hence the result follows from part (a). ∎

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Received 11 July 1996; final manuscript accepted 2 October 1996